PROPERTIES OF Z-TRANSFORM

✓ The z-Transform of a DT sequence is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The z-Transform pair is given by $\quad x[n] \longleftrightarrow X(z)$

The z-Transform obeys the following properties.

(1) LINEARITY: If $\quad x_1[n] \longleftrightarrow X_1(z)$ & $x_2(n) \longleftrightarrow X_2(z)$

$\text{ROC} = R_1 \qquad \text{ROC} = R_2$

then

$$a x_1[n] + b x_2[n] \longleftrightarrow a X_1(z) + b X_2(z)$$

with $\text{ROC} = R_1 \cap R_2$

Proof:

$$ZT\left[a x_1[n] + b x_2[n]\right] = \sum_{n=-\infty}^{\infty} \left[a x_1[n] + b x_2[n]\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a x_1[n] z^{-n} + \sum_{n=-\infty}^{\infty} b x_2[n] z^{-n}$$

$$= a X_1(z) + b X_2(z) \quad //$$

(2) TIME SHIFTING: If $\quad x[n] \longleftrightarrow X(z)$; $\text{ROC} = R_1$

then $\quad x[n-k] \longleftrightarrow z^{-k} X(z)$; $\text{ROC} = R_1$

Proof:

$$ZT\left[x[n-k]\right] = \sum_{n=-\infty}^{\infty} x[n-k] z^{-n}$$

; let $n-k = m$

$\Rightarrow n = m+k$

Also as $n \to \infty$, the $m \to \infty$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{-(m+k)}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot z^{-m} \cdot z^{-k}$$

$$= z^{-k} \cdot X(z) \quad //$$

(3) SCALING IN Z-DOMAIN: If $x[n] \longleftrightarrow X(z)$; ROC = R

then $a^n x[n] \longleftrightarrow X\left(\frac{z}{a}\right)$; ROC = $|a|R$

Proof: $$ZT\left[a^n x[n]\right] = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right) //$$

If ROC is $\alpha < |z| < \beta$, then the new ROC will be

$|a|\alpha < z < |a|\beta$.

(4) TIME REVERSAL: If $x[n] \longleftrightarrow X(z)$; ROC = R

then $x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$; ROC = $\frac{1}{R}$

Proof:

$$ZT\left[x[-n]\right] = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$; let $-n = m \Rightarrow$ As $n \to \infty$ the $m \to \infty$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{m} = \sum_{m=-\infty}^{\infty} x[m] \left(z^{-1}\right)^{-m}$$

$$= X\left(z^{-1}\right) \text{ or } X\left(\frac{1}{z}\right)$$

If ROC is $a < |z| < b$, then the new ROC will be

$a < \left|\frac{1}{z}\right| < b \Rightarrow \frac{1}{b} < |z| < \frac{1}{a}$ //

(5) DIFFERENTIATION IN Z-DOMAIN: If $x[n] \longleftrightarrow X(z)$; ROC = R_1

then $n \cdot x[n] \longleftrightarrow -z \cdot \frac{d}{dz}\left[X(z)\right]$; ROC = R_1

Proof: $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\frac{d}{dz}\left[X(z)\right] = \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{d}{dz}\left[z^{-n}\right] = \sum_{n=-\infty}^{\infty} x[n] \left(-n \cdot z^{-n-1}\right)$$

$$= -\frac{1}{z} \sum_{n=-\infty}^{\infty} n \cdot x[n] z^{-n}$$

$$-z \cdot \frac{d}{dz}\left[X(z)\right] = \sum_{n=-\infty}^{\infty} \left(n \cdot x[n]\right) \cdot z^{-n}$$

(or) $$-z\frac{d}{dz}\left[X(z)\right] = ZT\left[n \cdot x[n]\right] //$$

(6) CONVOLUTION PROPERTY

If $x_1[n] \longleftrightarrow X_1(z)$; ROC $= R_1$

$x_2[n] \longleftrightarrow X_2(z)$; ROC $= R_2$

then $x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$; ROC $= R_1 \cap R_2$

(Convolution in time domain $\longleftrightarrow$ Multiplication in z-domain)

Proof:

Convolution of two signals $x_1[n]$ and $x_2[n]$ is given by

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

Now the z-Transform of convolution is given by

$$ZT\left[x_1[n] * x_2[n]\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]\right] \cdot z^{-n}$$

Interchanging the order of summation

$$= \sum_{k=-\infty}^{\infty} x_1[k]\left[\sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n}\right]$$

$\Downarrow$ ZT of $x_2[n-k]$

If $x_2[n] \longleftrightarrow X_2(z)$

$x_2[n-k] \longleftrightarrow z^{-k} X_2(z)$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\, X_2(z)\, z^{-k} = X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]\, z^{-k}$$

$$= X_2(z) \cdot X_1(z)$$

$$= X_1(z) \cdot X_2(z) \;//$$