1. SINGULARITY FUNCTIONS

1.0 INTRODUCTION

Singularity functions are discontinuous functions or their derivatives are discontinuous.

A singularity is a point at which a function does not possess a derivative. In other words, a singularity function is discontinuous at its singular points. Hence a function that is described by polynomial in ***t*** is thus a singularity function. The commonly used singularity functions are:

- Step Function,
- Ramp Function, and
- Impulse Function.

At first, we take up the study of a unit-step function.

1.1 UNIT-STEP FUNCTION

The continuous-time unit-step function is defined as:

$$\left.\begin{aligned} \mathrm{u}(t) &= 1 \text{ for } \ t > 0 \\ &= 0 \text{ for } \ t < 0 \end{aligned}\right\}$$

Above equation defines what a unit-step function. The value of a unit-step function is one, for values of ***t*** > 0, and it is zero, for values of ***t*** < 0. It is undefined at ***t*** = 0. The unit-step function has a value between 0 and 1, at ***t*** = 0. The value of the unit-step function changes suddenly, at ***t*** = 0. Because of the step change in unit-step function at ***t***= 0, the value of derivative of unit-step function is infinite at ***t*** = 0. In other words, the unit-step function is discontinuous at ***t*** = 0. It can be seen that the derivative of unit-step function is zero at all instants, except ***t*** = 0.

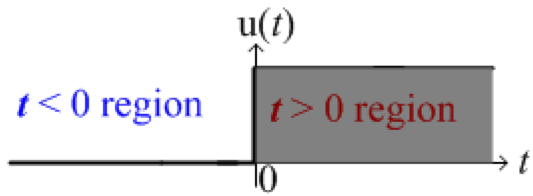


Fig.1. Plot of unit step function

The unit step function u(t) is represented as shown in Fig. 1. The unit step function is used widely in network theory and control theory. It can be seen that the unit step function has a discontinuity at $t = 0$ and is continuous for all other values of t .

1.1.1 Reflection operation on the unit-step function

It is easy to visualize how u(-t) would be. This function, u(-t) is reflected version of u(t) and is illustrated in Fig. 2.

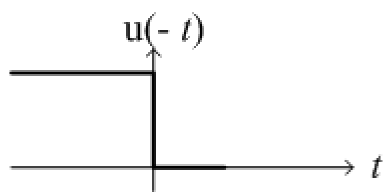


Fig. 2. Plot of u(-t)

Another example using the unit step function is shown in Fig. 3. This function is called the signum function and it is written as sgn(t).

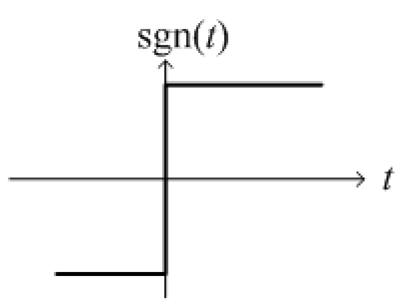


Fig.3. Plot of signum function

$$\operatorname{sgn}(t)=\begin{cases}1, & \text{for } t>0\\ 0 & \text{for } t=0\\ -1 & \text{for } t<0\end{cases}$$

The signum function is often not used in network theory, but it is used in communication and control theory. It is expressed in terms of unit step functions as indicated below

(or) **sgn(t)= -1+2u(t)**

(or) **sgn(t)= u(t)-u(-t)**

1.1.2 Shifting operation on the unit-step function

A shifted time function is displayed in Fig. 4.

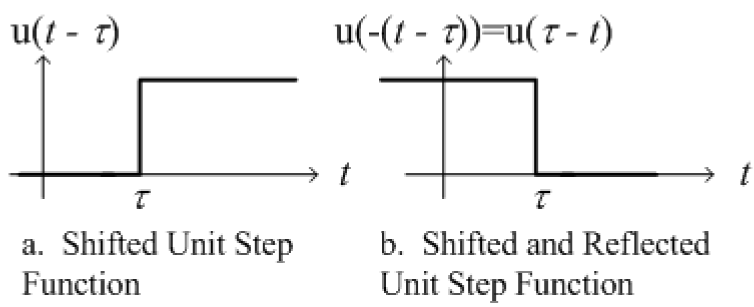


Fig.4. Shifting and Folding operation on step function

The shifted function can be expressed as shown below:

$$\left.\begin{aligned}u(t-\tau)&=1 \text{ for } t>\tau\\ &=0 \text{ for } t<\tau\end{aligned}\right\}$$

In a shifted unit step function, defined above equation , the step change occurs at ***t*** = τ, whereas the step change occurs at ***t*** = 0 for the unit step function defined by u(t). It can be seen that the shifted unit step function is obtained by shifting the unit step function to the right by τ seconds. It is seen from equation that when $t > \tau$, the argument of the shifted unit step function is positive and then the function has unit value. When the argument of the shifted unit step function is negative, the function has zero value. It can be seen that the argument of the shifted unit step function is negative for $t < \tau$.

Synthesis of a signal

It is possible to use singularity functions to generate or synthesize different signals. An example is shown below to show how a rectangular pulse signal can be visualized as the combination of two step functions.

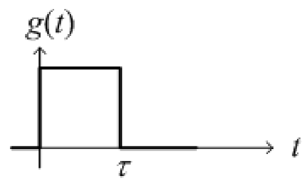


Fig. 5. A rectangular pulse function

A rectangular pulse function of unit amplitude, illustrated in Fig. 5, is obtained as the combination of a unit step function and a shifted step function with a magnitude of - 1.

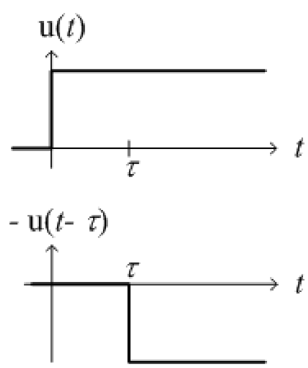


Fig.6. Signals used for synthesis of a pulse signal

When the two signals shown in Fig. 6 are added, we get the rectangular pulse shown in Fig. 5. We get

$g(t)=u(t)-u(t-\tau)$

A system may receive a single rectangular pulse, as shown in Fig. 5. If this pulse repeats itself after a fixed period, then the resulting signal is a square-wave periodic signal.

RAMP FUNCTION

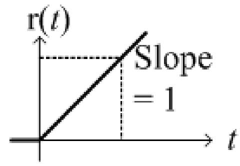


Fig.7. A ramp function

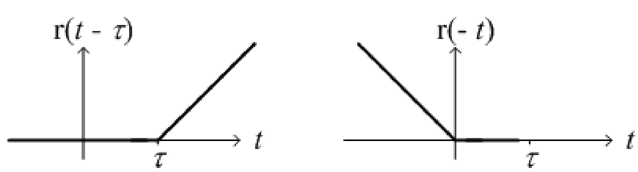
The ramp function is illustrated in Fig. 7. It can be defined as follows:

$$\left.\begin{aligned} r(t) &= t \quad \text{for } t \geq 0 \\ &= 0 \quad \text{for } t < 0 \end{aligned}\right\}$$

The ramp function has zero value in the range defined by ***t*** **<** 0. When ***t*** > 0, the ramp function increases linearly with time. The unit-step function and the ramp function are related. We can define the unit-step function, as the derivative of the ramp function,. Alternatively, we can state that the ramp function is the integral of the unit-step function.

$$u(t) = \frac{dr(t)}{dt}$$

$$r(t) = \int_{-\infty}^{t} u(t)dt = \int_{-\infty}^{t} 1.dt = t.u(t)$$



Shifted ramp function

The reflected ramp Function

Fig.8. Shifted ramp, reflected ramp

The effect of operations on the independent variable of the ramp function is shown by the sketches in Fig. 8. The plots of the shifted ramp function and the reflected ramp function are displayed.

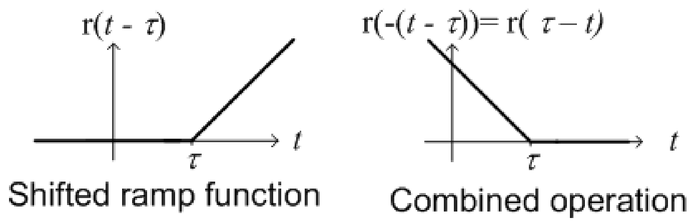


Fig. 9. Operations on ramp signal

It is possible to shift the ramp function and then reflect it, as shown in Fig. 9. The ramp function is a signal generated by some electronic circuits. With additional electronic circuitry, it is possible to generate saw-tooth waveform displayed in Fig. 10 . Such a signal is used in a cathode-ray oscilloscope (CRO) as the timing signal. Such a signal is used in a TV also for horizontal and vertical scanning.

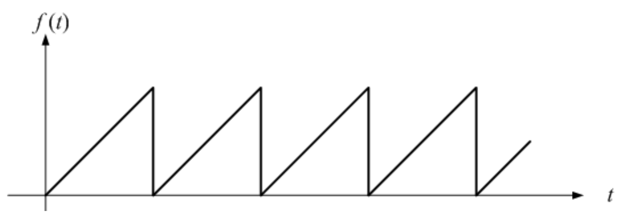


Fig.10. Saw-tooth waveform, used as sweep signal in CROs

IMPULSE FUNCTION

The unit impulse function, designated $\delta(t)$, is also called the Dirac delta function. Its use in network theory, control theory and signal theory is widespread and it is important because of its properties and the insight it offers about the network to which it is applied.

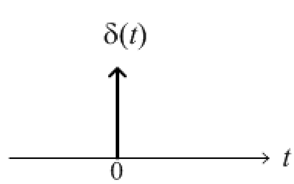


Fig.11. Impulse or Dirac delta function

The impulse function is displayed, as shown in Fig. 11. The impulse function is related to the unit-step function. It is the integral of the impulse function.

$$u(t) = \int_{-\infty}^{t} \delta(t)dt$$

Alternatively, the unit impulse function is defined as the derivative of the unit step function, as expressed below

$$\delta(t) = \frac{du(t)}{dt}$$

The ideal impulse function is represented by a spike at the origin as shown in Fig. 11.

The impulse has the following properties.

1. $\delta(0) \to \infty$
2. $\delta(t) = 0$ if $t \neq 0$
3. $\int_{-\infty}^{\infty} \delta(t).dt = 1$ and
4. $\delta(t) = \delta(-t)$, that is, $\delta(t)$ is an even function.

Sifting/Sampling Property

Given a continuous time-function $f(t)$, sampling property is defined as:

$$\int_{-\infty}^{\infty} f(t).\delta(t-a).dt \equiv f(a)$$

This property is also known as the sifting property. Since $\delta(t - a)=0$ when $t \neq a$, the above integral can be expressed to be:

$$\because \delta(t-a)=0, \text{ when } t \neq a$$

$$\int_{-\infty}^{\infty} f(t).\delta(t-a).dt = f(a)\int_{-\infty}^{\infty}\delta(t-a).dt = f(a)$$

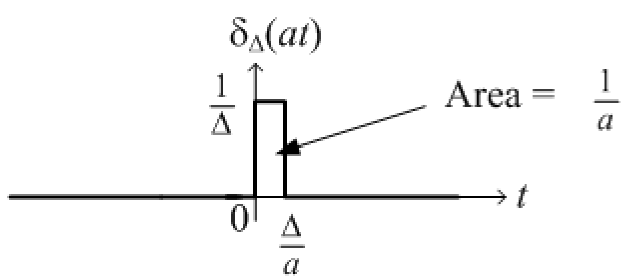
Sampling property shows how a function can be sampled at a given instant.

$$f(t).\delta(t-a) \equiv f(a).\delta(t-a)$$

Since the impulse function has value only at $t = a$, the value of $f(t)$ when $t \neq a$ is not important. We call this property as the sifting property.

Time Scaling Property

$$\delta(at) \quad = \underset{\Delta \to 0}{Limit}\ \delta_\Delta(at) = \frac{1}{a} \times \delta(t)$$



The impulse response is significant since it reveals the nature of the system. The poles of impulse response are the poles of the system. We can use convolution integral to obtain the response of a system to any input. To apply the convolution integral, we make use of the impulse response.

SYNTHESIS OF WAVEFORMS

It is possible to synthesize waveforms using singularity functions. An example has already been presented, wherein a pulse function has been generated as the sum of step functions.