Causality & Memory:

- A system whose present output (response) requires future inputs are termed as Non-Causal
- what makes the LCCDE non-causal.
  - If the least delayed output is $y[n]$ then terms like $x[n+k]$; $k>0$ will make it NC

Ex:

- $y[n] - 3y[n-2] = x[n] \rightarrow$ causal
- $y[n] - 3y[n-2] = x[n+2] \rightarrow$ Noncausal
- $y[n+1] - y[n] = x[n+1] \rightarrow$ causal
- $y[n+2] - y[n+1] - y[n] = x[n+3] \rightarrow$ Noncausal.

Memory:
- Static or Instantaneous
- Dynamic

If the response of the system at $n=n_0$ depends only on the input at $n=n_0$ and not at any other times (past & future), the system is called static or instantaneous.

Ex:

$$y[n] = 2x[n] \rightarrow \text{static}$$

$$y[n] = 2x[n-1] \rightarrow \text{dynamic}$$

$$y[n-1] = 2x[n-1] \rightarrow \text{static}$$

$$y[n-1] = 2x[n-2] \rightarrow \text{Dynamic}$$

✓ **Causal LTI systems*:** (condition on $h[n]$)

If $h[n] = 0$ for $n < 0$

the system is <u>causal LTI</u> system

✓ **Stable LTI systems*:** (condition on LTI system)

~~A system~~

A DT-LTI system is stable if and only if the impulse response $h[n]$ is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Proof

Proof: A system is Bounded input - Bounded output (BIBO) stable if the o/p is guaranteed to be bounded for every bounded input.

i.e, if $|x[n]| \le M_x < \infty$,

then $|y[n]| \le M_y < \infty$

We know output of an LTI system is convolution sum of input $x[n]$ and impulse response $h[n]$

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$\because \quad |a+b| \leq |a| + |b|$

$$y[n] \leq \sum_{k=-\infty}^{\infty} \left| h[k]\, x[n-k] \right|$$

$\because \quad |ab| = |a||b|$

$$y[n] \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

as $|x[k]| \leq M_x < \infty$

the $|x[n-k]| \leq M_x$.

Hence $\quad y[n] \leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$

Hence output is bounded, provided impulses response $h[n]$ is absolutely summable.

$\therefore$ An LTI system is stable if and only if, its impulse response is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Ex :

$h[n] = 2^n u[-n]$: not memoryless, not causal, stable

$h[n] = e^{2n} u[n-1]$; not memoryless, causal, not stable

$h[n] = \left(\frac{1}{2}\right)^n u[n]$; not memoryless, causal, stable

$h[n] = \cos\left(\frac{\pi}{2} n\right) u[n+3]$;

not memoryless, not causal, not stable.

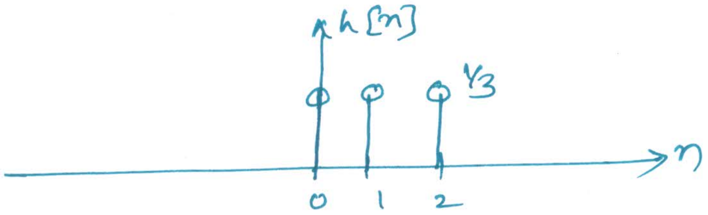
Ex: Show that a 3-sample moving-average system is stable

Sol:
$$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$$

Find $h[n]$:

if $x[n] = \delta[n]$, then $y[n] = h[n]$

$$\therefore h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$



$$\therefore h[n] = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3} \}$$

Stability:
$$S = \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{2} |h[k]| = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$S < \infty$; $h[n]$ is absolutely summable.

Hence stable system

Ex: Consider the DT system $y[n] = r^n x[n]$

Comment on its stability for $r > 1$

Ans: Unstable

Sol: if $x[n] = \delta[n]$; $y[n] = h[n]$

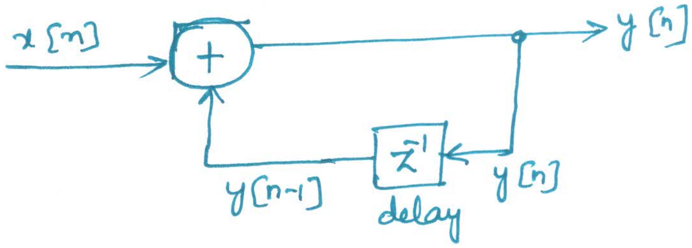
$$\therefore h[n] = r^n \delta[n]$$

$n=0$; $h[0] = r^0 \delta[0] = 1$

$h[1] = r^1 \delta[1] = 0$

Ex: Check the stability of the DT system

$y[n] = a\,y[n-1] + x[n]$; Assume initially relaxed.



Sol: $y[n] = a\,y[n-1] + x[n]$

if $x[n] = \delta[n]$ then $y[n] = h[n]$

$\therefore\ h[n] = a\,h[n-1] + \delta[n]$

$n=0$; $h[0] = a\,h[-1] + \delta[0] = 0 + 1 = 1$

$n=1$; $h[1] = a\,h[0] + \delta[1] = a \cdot 1 + 0 = a$

$n=2$; $h[2] = a\,[h[1]] + \delta[2] = a \cdot [a] + 0 = a^2$

$n=3$; $h[3] = a\,h[2] + \delta[3] = a[a^2] + 0 = a^3$

$\vdots$

$h[n] = a^n$; $n \geq 0$

$\therefore\ h[n] = a^n u[n].$

Stability: $S = \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |a^k| = \frac{1}{1-|a|}$

It converges only if $|a| < 1$

$\therefore$ The given system is stable only when $|a| < 1$

Note

Ex: $y[n] = 0.5\,y[n-1] + x[n]$ is stable system

$y[n] = 2\,y[n-1] + x[n]$ is an unstable system.

✓ Investigate the causality and stability of the following systems:

① $h[n] = 2^n u[n-1]$; It is causal.

$$S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=1}^{\infty} 2^n = 2+4+8+\cdots = \infty$$

Hence Unstable

② $y[n] = 2x[n+1] + 3x[n] - x[n-1]$

↳ Noncausal

$h[n] = 2\delta[n+1] + 3\delta[n] - \delta[n-1]$

$\therefore h[n] = \{2, \underset{\uparrow}{3}, -1\}$

$\therefore S = \sum_{n} |h[n]| = 2+3+(1) = \underline{\underline{6}}$ Hence stable

③ $h[n] = (-0.5)^n u[n]$

causal

$$S = \sum_{n} |h[n]| = \sum_{n\geq 0}^{\infty} (0.5)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

Hence Stable

④ $h[n] = \{3, 2, \underset{\uparrow}{1}, 1, 2\}$

Noncausal

$S = \sum h[n] = 3+2+1+1+2 = 9$; Hence stable

⑤ $h[n] = (-0.5)^n u[-n]$

non-causal

$$S = \sum_{n=-\infty}^{\infty} (-0.5)^n u[-n] = \sum_{n=-\infty}^{0} \left(-\frac{1}{2}\right)^n = \sum_{n\geq 0}^{\infty}$$

stable

⑥ $h[n] = (0.5)^{|n|}$

Non causal and stable

(two-sided decaying exponential)