Ex: Consider the system $y[n] - 0.5\,y[n-1] = x[n]$
Find the response of this system for the i/p $x[n] = (0.5)^n u[n]$

Sol: To find response to $x[n]$, let us use convolution sum.

$$y[n] = x[n] * h[n]$$

(1) Find $h[n]$:

Given $y[n] - 0.5\,y[n-1] = x[n]$
$y[n] = 0.5\,y[n-1] + x[n]$

if $x[n] = \delta[n]$, then $y[n] = h[n]$

$$h[n] = 0.5\,h[n-1] + \delta[n]$$

$n=0$; $h[0] = 0.5\,h[-1] + \delta[0] = 0 + 1 = 1$

$n=1$; $h[1] = 0.5\,h[0] + \delta[1] = 0.5 + 0 = 0.5$

$n=2$; $h[2] = 0.5\,h[1] + \delta[2] = (0.5)(0.5) + 0 = (0.5)^2$

⋮

$$h[n] = (0.5)^n u[n]$$

(2) $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k].$$

$x[k] = (0.5)^k u[k]$; $x[k] = 0$ for $k < 0$

$h[n-k] = (0.5)^{n-k} u[n-k]$; $h[n-k] = 0$ for $k > n$

limits of summation $k=0, k=n$

$$y[n] = \sum_{n=-\infty}^{\infty} (0.5)^k u[k]\,(0.5)^{n-k} u[n-k] = \sum_{k=0}^{n} (0.5)^k (0.5)^{n-k}$$

$$= (0.5)^n \sum_{n=0}^{n} 1 = (n+1)(0.5)^n u[n]$$

Ex : Find $y[n] = u[n+3] * u[n-3]$

Sol:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$x[k] = u[k+3]$; $x[k] = 0$; for $k < -3$

$h[n-k] = u[n-k-3]$; $h[n-k] = 0$; for $k > n-3$

$\downarrow$ $-k = 3-n \Rightarrow k = n-3$

Hence overlapping limits of summation are $k = -3$ to $k = n-3$

$$\therefore y[n] = \sum_{k=-\infty}^{\infty} u[k+3]\,u[n-k-3] = \sum_{k=-3}^{n-3} 1 = \underbrace{1+1+1+1+1+1+1}_{(n+1)}$$

$$y[n] = (n+1) \;;\; n \geq 0$$

Ex : $y[n] = \beta^n u[n] * u[n-3]$ with $|\beta| < 1$

Sol: $y[n] = \beta^n x[n] * h[n]$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

~~$h[k] = u[k-3]$; $h[k] = 0$; for $k < 3$~~

~~$x[n-k] = \beta^{n-k} u[n-k]$; $x[n-k] = 0$ for $k > n$~~ | ~~$k = 2$ to $k = n$~~

~~$\therefore y[n] = \sum_{k=-\infty}^{\infty} u[k-3]\,\beta^{n-k} u[n-k] = \sum_{k=2}^{n} \beta^{n-k}$~~

~~$y[n] = \beta^n \sum_{k=2}^{n} \beta^{-k}$~~

$h[k] = \beta^k u[k]$; $h[k] = 0$ for $k < 0$

$x[n-k] = u[n-k-3]$; $x[n-k] = 0$ for $k > n-3$

$\hookrightarrow k = n-3$

limits $k = 0$ to $k = n-3$

$$y[n] = \sum_{k=-\infty}^{\infty} \cancel{\alpha^n} \beta^k u[k]\, u[n-k] = \sum_{k=0}^{n-3} \beta^k$$

$$y[n] = \frac{1-\beta^{(n-3+1)}}{1-\beta}$$

$$\boxed{y[n] = \frac{1-\beta^{n-2}}{1-\beta}} \qquad \underline{\underline{n \geq 3}}.$$

<u>Ex :</u> $y[n] = \beta^n u[n] * \alpha^n u[n-10]$: $|\beta|<1$, $|\alpha|<1$

<u>Sol:</u> $y[n] = x[n] * h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

here $x[k] = \beta^k u[k]$; $x[k]=0$ for $k<0$

$h[n-k] = \alpha^{n-k} u[n-k-10]$; $h[n-k]=0$ for $k>(n-10)$

$\hookrightarrow k = n-10$

$\therefore$ limits of summation are $k=0$ to $k=n-10$

$$y[n] = \sum_{k=-\infty}^{\infty} \beta^k u[k]\, \alpha^{n-k} u[n-k-10] = \sum_{k=0}^{n-10} \beta^k . \alpha^{n-k}$$

$$= \alpha^n \sum_{k=0}^{n-10} \left(\frac{\beta}{\alpha}\right)^k = \alpha^n \left[\frac{1-\left(\frac{\beta}{\alpha}\right)^{n-10+1}}{1-(\beta/\alpha)}\right]$$

$$= \alpha^n \left[\frac{1-(\beta/\alpha)^{n-9}}{1-(\beta/\alpha)}\right] ; \alpha \neq \beta$$

$$= \alpha^n (n-9) \quad ; \text{if } \alpha = \beta.$$

Ex: The I/O difference equation of first-order recursive system is given by

$$y[n] - a\,y[n-1] = x[n]$$

Let the input is given by $x[n] = b^n u[n+4]$

Find its output using convolution sum.

Sol: (i) Find $h[n]$:

$$h[n] = a^n u[n]$$

(ii) $$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

here $x[k] = b^k u[k+4]$; $x[k] = 0$ for $k < -4$

$h[n-k] = a^{n-k} u[n-k]$; $h[n-k] = 0$ for $k > n$

$\therefore$ limits of summation are $\underline{k = -4 \text{ to } k = n}$

$$y[n] = \sum_{k=-\infty}^{\infty} b^k u[k+4]\, a^{n-k} u[n-k] = \sum_{k=-4}^{n} b^k . a^{n-k}.$$

$$= a^n \sum_{k=-4}^{n} \left(\frac{b}{a}\right)^k = \cancel{a^n \sum_{k=0}^{n+4} \left(\frac{b}{a}\right)^k}$$

$$= \cancel{a^n \frac{1-\left(\frac{b}{a}\right)^{n+5}}{1-\frac{b}{a}}}$$

let $m = k+4$, then

$$y[n] = a^n \sum_{m=0}^{n+4} \left(\frac{b}{a}\right)^{m-4}$$

$$y[n] = a^n \left(\frac{b}{a}\right)^4 \sum_{m=0}^{n+4} \left(\frac{b}{a}\right)^m = a^n \left[\frac{b}{a}\right]^4 \left[\frac{1-\left(\frac{b}{a}\right)^{n+5}}{1-\frac{b}{a}}\right]$$

$\underline{-4 \le n}$