Some Problems on Duality Property of FT

if $x(t) \longleftrightarrow X(\omega)$

then $X(t) \longleftrightarrow 2\pi x(-\omega)$

✓ find the FT of the signal $g(t) = \frac{1}{1+jt}$.

Sol: Think a signal whose FT has a similar expression as that of $g(t)$.

we know $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$

with $a=1$; $e^{-t}u(t) \longleftrightarrow \frac{1}{1+j\omega}$

$\Rightarrow$ say $x(t) \longleftrightarrow X(\omega)$

i.e, $x(t) = e^{-t}u(t)$; $X(\omega) = \frac{1}{1+j\omega}$

Change of variable:

replace t with $-\omega$; $x(-\omega) = e^{\omega}u(-\omega)$; Replace ω with t $X(t) = \frac{1}{1+jt}$

now invoking the Duality property of FT

if $x(t) \longleftrightarrow X(\omega)$

then $X(t) \longleftrightarrow 2\pi x(-\omega)$

$$\frac{1}{1+jt} \longleftrightarrow 2\pi\left[e^{\omega}\cdot u(-\omega)\right]$$

$$\therefore FT\left[\frac{1}{1+jt}\right] = 2\pi\cdot e^{\omega}u(-\omega)$$

✓ Find the FT of the signal $x(t) = \dfrac{1}{1+t^2}$

Sol: Lets solve this by making use of Duality property of FT.

∴ Think a signal whose FT has similar expression as that of $x(t)$.

we know that $$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$$

with $a=1$; $$e^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2}$$

$$\frac{1}{2}\cdot e^{-|t|} \longleftrightarrow \frac{1}{1+\omega^2}$$

Say $$g(t) \longleftrightarrow \frac{1}{1+\omega^2}$$

~~change the variables;~~

$$\therefore g(t) = \frac{1}{2} e^{-|t|} \quad \text{and} \quad G(\omega) = \frac{1}{1+\omega^2}$$

change of variables:

Replace t with ω;
$$g(\omega) = \frac{1}{2} e^{-|\omega|}$$

Replace ω with t;
$$G(t) = \frac{1}{1+t^2}$$

also $$g(-\omega) = \frac{1}{2} e^{-|\omega|}$$

Using Duality property of FT, which states that

if $x(t) \longleftrightarrow X(\omega)$

then $X(t) \longleftrightarrow 2\pi\, x(-\omega)$

Here $$G(t) \longleftrightarrow 2\pi g(-\omega)$$

$$\frac{1}{1+t^2} \longleftrightarrow 2\pi \cdot \frac{1}{2} e^{-|\omega|} = \pi e^{-|\omega|}$$

Hence $$FT\left[\frac{1}{1+t^2}\right] = \pi e^{-|\omega|}$$

Find the FT of the signal $g(t) = \frac{1}{\pi t}$

Sol: Let's solve this using duality property of FT

Think a signal whose FT has a similar expression as that of $g(t)$

we know $\quad \text{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$

manipulating on both sides to get $\frac{1}{\pi\omega}$ on RHS,

$$\frac{j}{2\pi}\,\text{sgn}(t) \longleftrightarrow \frac{j}{2\pi}\left[\frac{2}{j\omega}\right]$$

$$\frac{j}{2\pi}\,\text{sgn}(t) \longleftrightarrow \frac{1}{\pi\omega}$$

say $x(t) = \frac{j}{2\pi}\,\text{sgn}(t)$ and $X(\omega) = \frac{1}{\pi\omega}$

<u>Change of variables</u>

Replacing t by ω, we have

$$x(\omega) = \frac{j}{2\pi}\,\text{sgn}(\omega)$$

$$x(-\omega) = \frac{j}{2\pi}\,\text{sgn}(-\omega)$$

Replacing ω by t, we have

$$X(t) = \frac{1}{\pi t}$$

Using Duality property of FT, which states that

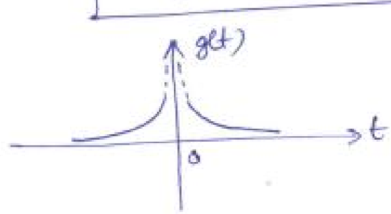
if $x(t) \longleftrightarrow X(\omega)$

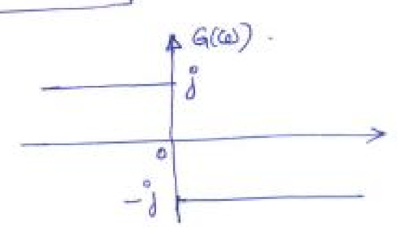
then $X(t) \longleftrightarrow 2\pi\, x(-\omega)$

here, $\quad \frac{1}{\pi t} \longleftrightarrow 2\pi \cdot \frac{j}{2\pi}\,\text{sgn}(-\omega)$

$$\frac{1}{\pi t} \longleftrightarrow -j\,\text{sgn}(\omega)$$

$$\boxed{F\left[\frac{1}{\pi t}\right] = -j\,\text{sgn}(\omega)}$$





✓ Find the inverse FT of $u(\omega)$

Sol: Let's solve using duality property of FT

Think $u(t)$ and its FT

$$u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

using time-reversal property

$$u(-t) \longleftrightarrow \frac{1}{|-1|}\pi\delta\left(\frac{\omega}{-1}\right) + \frac{1}{j\left(\frac{\omega}{-1}\right)}$$

$$u(-t) \longleftrightarrow \pi\delta(-\omega) - \frac{1}{j\omega}$$

$$u(-t) \longleftrightarrow \pi\delta(\omega) - \frac{1}{j\omega}$$

$$\therefore \quad x(t) \longleftrightarrow X(\omega)$$

i.e, $x(t) = u(-t)$ and $X(\omega) = \pi\delta(\omega) - \frac{1}{j\omega}$

Change of variable

Replacing t by ω,

$$x(\omega) = u(-\omega)$$

and $x(-\omega) = u(\omega)$

Replacing ω by t

$$X(t) = \pi\delta(t) - \frac{1}{jt}$$

Using duality property of FT, if $x(t) \longleftrightarrow X(\omega)$

then $X(t) \longleftrightarrow 2\pi x(-\omega)$

Substituting, for $X(t) \longleftrightarrow 2\pi x(-\omega)$

$$\left(\cancel{2}\pi\delta(t) - \frac{1}{jt}\right) \longleftrightarrow 2\pi \cdot u(\omega)$$

$$\frac{1}{2\pi}\left[\cancel{2}\pi\delta(t) - \frac{1}{jt}\right] \longleftrightarrow u(\omega)$$

$$\therefore \quad FT^{-1}\left[u(\omega)\right] = \frac{1}{2}\delta(t) - \frac{1}{j2\pi t}$$

Ex: Find the IFT of $X(jw) = \pi e^{-|w|}$

Sol: Let's solve using duality property. Think a signal having a similar expression in time domain

we know $FT\left[e^{-a|t|}\right] = \frac{2a}{a^2+\omega^2}$

or $FT\left[e^{-|t|}\right] = \frac{2}{1+\omega^2}$

$x(t) \longleftrightarrow X(\omega)$ | Duality property

then $X(t) \longleftrightarrow x(-\omega)\cdot 2\pi$ |

$\therefore \quad FT\left[X(t)\right] = 2\pi \cdot x(-\omega).$

$FT\left[X(t)\right] = 2\pi x(-\omega) \quad \text{—— (1)}$

Here $\quad X(\omega) = \frac{2}{1+\omega^2} \Rightarrow X(t) = \frac{2}{1+t^2}$

$x(t) = e^{-|t|} \Rightarrow x(-\omega) = e^{-|-\omega|} = e^{-|\omega|}$

(1) $\Rightarrow \quad FT\left[\frac{2}{1+t^2}\right] = 2\pi \cdot e^{-|\omega|}$

$\therefore \quad \frac{2}{1+t^2} = IFT\left[2\pi \cdot e^{-|\omega|}\right]$

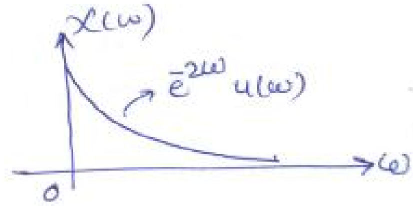
$\frac{2}{1+t^2} = 2\,.\, IFT\left[\pi e^{-|\omega|}\right]$

$\underline{IFT\left[\pi \cdot e^{-|\omega|}\right] = \frac{1}{1+t^2}}$

Ex: Find the IFT of $X(\omega) = e^{-2\omega} u(\omega)$

Sol:

$$X(\omega) = e^{-2\omega} u(\omega)$$



Using Duality property

Think of a signal having similar expression in time domain that can be $e^{-at}u(t)$.

$$\therefore \; FT\left[e^{-at}u(t)\right] = \frac{1}{a+j\omega}$$

$$FT\left[e^{-2t}u(t)\right] = \frac{1}{2+j\omega}.$$

$$x(t) \longleftrightarrow X(\omega)$$

then $X(t) \longleftrightarrow 2\pi\, x(-\omega)$. | Duality property

Here $X(\omega) = \dfrac{1}{2+j\omega} \Rightarrow X(t) = \dfrac{1}{2+jt}$

and $x(t) = e^{-2t}u(t) \Rightarrow x(\omega) = e^{-2\omega}u(\omega)$

$$x(-\omega) = e^{2\omega}u(-\omega)$$

$$\therefore FT\left[X(t)\right] = 2\pi\, x(-\omega).$$

$$FT\left[\frac{1}{2+jt}\right] = 2\pi e^{2\omega}u(-\omega)$$

But we need IFT of $e^{-2\omega}u(\omega)$.

using Time-reversal property $x(-t) \longleftrightarrow X(-\omega)$

$$\therefore FT\left[\frac{1}{2-jt}\right] = 2\pi e^{-2\omega}u(\omega).$$

$$\text{or } IFT\left[e^{-2\omega}\cdot u(\omega)\right] = \frac{1}{2\pi} \times \frac{1}{2-jt}$$

$$IFT\left[e^{-2\omega}\cdot u(\omega)\right] = \frac{1}{2\pi}\left[\frac{1}{2-jt}\right]$$