Ex: A causal DT is described by the following LCCDE

$$y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n]$$

(a) Find the transfer function $H(z)$ for causal system
(b) Find and plot poles and zeros
(c) Comment on the stability of the system
(d) Find the impulse response of the system
(e) Find the step response of the system

Sol:

$$y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n]$$

Taking z-Transform, we obtain

If $x[n] \longleftrightarrow X(z)$
then $x[n-k] \longleftrightarrow z^{-k}X(z)$

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Thus, the transfer function (or system function $H(z) = \frac{Y(z)}{X(z)}$) is

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

(or) $$\boxed{H(z) = \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}}}$$

(b) Poles: Roots of Denominator

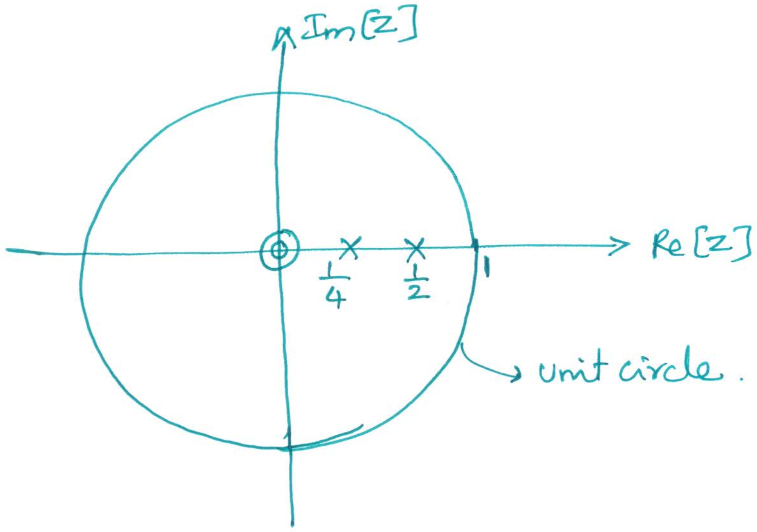
Zeros: Roots of Numerator

H(z) can be written as $H(z) = \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$

Hence Poles are $z_1 = \frac{1}{4}$ and $z_2 = \frac{1}{2}$

Zeros: Second order zero at $z=0$

The pole-zero plot is shown below



(C)
Stability : For a causal system : A causal system is said to be stable, if all its poles lie inside unit circle in Z-plane.

As all the poles ($z_1 = \frac{1}{4}$ & $z_2 = \frac{1}{2}$) are inside unit circle, the given system is causal.

(d) Impulse response $h[n]$ of a system is IZT of transfer function.

$$h[n] = IZT\left[H(z)\right]$$

Here $H(z) = \dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$

Finding IZT using partial fractions

$$\frac{H(z)}{z} = \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)} = \frac{A}{\left(z-\frac{1}{4}\right)} + \frac{B}{\left(z-\frac{1}{2}\right)}$$

here $A = \left[\left(z-\frac{1}{4}\right)\frac{H(z)}{z}\right]_{z=\frac{1}{4}} = \left[\frac{z^2}{z-\frac{1}{2}}\right]_{z=\frac{1}{4}} = -1$

$$B = \left[\left(z-\frac{1}{2}\right)\frac{H(z)}{z}\right]_{z=\frac{1}{2}} = \left[\frac{z^2}{z-\frac{1}{4}}\right]_{z=\frac{1}{2}} = 2$$

$$\frac{H(z)}{z} = -\left[\frac{1}{z-\frac{1}{4}}\right] + \left[\frac{2}{z-\frac{1}{2}}\right]$$

$$H(z) = -\left[\frac{z}{z-\frac{1}{4}}\right] + \left[\frac{2z}{z-\frac{1}{2}}\right]$$

$$H(z) = -\left[\frac{1}{1-\frac{1}{4}z^{-1}}\right] + 2\left[\frac{1}{1-\frac{1}{2}z^{-1}}\right]$$

For causal signal, we have

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \; ; \; |z| > |a|.$$

$\therefore$

$$\frac{1}{1-\frac{1}{4}z^{-1}} \longleftrightarrow \left(\frac{1}{4}\right)^n u[n] \; ; \; |z| > \frac{1}{4}$$

and

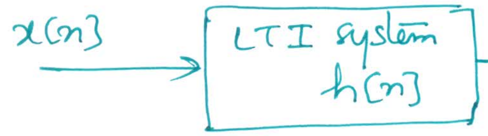
$$\frac{1}{1-\frac{1}{2}z^{-1}} \longleftrightarrow \left(\frac{1}{2}\right)^n u[n] \; ; \; |z| > \frac{1}{2}$$

$\therefore$ the IZT is given by

$$h[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

(e) <u>step response</u>: input $x[n] = \text{step} = u[n]$

$y[n] = \text{step response} = ?$



$\rightarrow y[n] = x[n] * h[n].$

Taking ZT convolution property

$$Y(z) = X(z)\,H(z).$$

Here $x[n] = u[n]$

$$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

and $$H(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\therefore Y(z) = X(z)\cdot H(z) = \frac{z^3}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})}$$

Hence $y[n]$ is obtained by taking inverse ZT of $Y(Z)$.

Lets use partial fraction method to find IZT

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{(z-1)} + \frac{B}{(z-\frac{1}{2})} + \frac{C}{(z-\frac{1}{4})}$$

here $$A = \left[(z-1)\frac{Y(z)}{z}\right]_{z=1} = \left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}\right]_{z=1} = 8/3$$

$$B = \left[\left(z-\frac{1}{2}\right)\frac{Y(z)}{z}\right]_{z=\frac{1}{2}} = \left[\frac{z^2}{(z-1)(z-\frac{1}{4})}\right]_{z=\frac{1}{2}} = -2$$

$$C = \left[\left(z-\frac{1}{4}\right)\frac{Y(z)}{z}\right]_{z=\frac{1}{4}} = \left[\frac{z^2}{(z-1)(z-\frac{1}{2})}\right]_{z=\frac{1}{4}} = 1/3$$

$$\therefore \frac{Y(z)}{z} = \frac{8}{3}\left[\frac{1}{z-1}\right] - 2\left[\frac{1}{z-\frac{1}{2}}\right] + \frac{1}{3}\left[\frac{1}{z-\frac{1}{4}}\right].$$

$$Y(z) = \frac{8}{3}\left[\frac{z}{z-1}\right] - 2\left[\frac{z}{z-\frac{1}{2}}\right] + \frac{1}{3}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$Y(z) = \frac{8}{3}\left[\frac{1}{1-z^{-1}}\right] - 2\left[\frac{1}{1-\frac{1}{2}z^{-1}}\right] + \frac{1}{3}\left[\frac{1}{1-\frac{1}{4}z^{-1}}\right]$$

we know $$\frac{1}{1-az^{-1}} \longleftrightarrow a^n u[n]$$

Hence Taking IZT, we have

$$y[n] = \frac{8}{3}(1)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{4}\right)^n u[n]$$

The step response $y[n] = \frac{8}{3}u[n] - 2\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{4}\right)^n u[n]$ //.

Ex: consider a DT system with following LCCDE

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] + \frac{1}{3}x[n-1]$$

Find unit sample response of the system.

Hint: Unit sample response means impulse response.

(1) Find $H(z)$: Ans $H(z) = \dfrac{z\left(z+\frac{1}{3}\right)}{\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)}$

(2) Find IZT of $H(z)$ to get $h[n]$

Ans: $h[n] = \frac{10}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{7}{3}\left(\frac{1}{4}\right)^n u[n]$ //

Ex: Find

(a) Impulse response

(b) step response of a system defined by the following LCCDE.

$$y[n] = y[n-1] + 0.5y[n-2] + x[n] + x[n-1]$$

Hints: First obtain transfer function $H(z) = \dfrac{Y(z)}{X(z)}$

Ans: $H(z) = \dfrac{z(z+1)}{(z^2 - z - 0.5)}$

(a) impulse response $h[n] = IZT\left[H(z)\right]$.

$$h[n] = 1.366\,(1.366)^n u[n] - 0.366\,(-0.366)^n u[n]$$

(b) step input $X(z) = \dfrac{1}{1-z^{-1}} = \dfrac{z}{z-1}$

$$\therefore Y(z) = X(z)H(z) = \left[\frac{z}{z-1}\right]\left[\frac{z(z+1)}{z^2 - z - 0.5}\right]$$

Taking IZT, we get stepresponse

$$y[n] = 5(1.366)^n u[n] + 0.1\,(-0.366)^n u[n] + 4u[n]$$ //