SIGNAL TRANSMISSION THROUGH LTI SYSTEMS

Let $h(t)$ = impulse response of an LTI system

$x(t)$ = input

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Then output $y(t) = x(t) * h(t)$

Applying FT on both sides and using convolution property

$$FT[y(t)] = FT[x(t) * h(t)]$$

$$Y(\omega) = X(\omega) H(\omega) \quad \text{—— (1)}$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

This is called Frequency Response $H(\omega)$ of the LTI system.

$|H(\omega)|$ is called magnitude response and

$\angle H(\omega)$ is called phase response of the system

(1) $\Rightarrow$ $\cancel{|Y(\omega)| e^{j\angle Y(\omega)} = |X(\omega)| |H(\omega)|}$

$$|Y(\omega)| e^{j\angle Y(\omega)} = |X(\omega)| e^{j\angle X(\omega)} \cdot |H(\omega)| e^{j\angle H(\omega)}$$

$$|Y(\omega)| e^{j\angle Y(\omega)} = |X(\omega)||H(\omega)| e^{j[\angle X(\omega) + \angle H(\omega)]}$$

$$\therefore \quad |Y(\omega)| = |X(\omega)||H(\omega)| \quad \text{—— (2)}$$

$$\text{and} \quad \angle Y(\omega) = \angle X(\omega) + \angle H(\omega). \quad \text{—— (3)}$$

(2) $\Rightarrow$ The input signal magnitude spectrum $|X(\omega)|$ is changed to $|X(\omega)||H(\omega)|$

(3) $\Rightarrow$ The input signal phase spectrum $\angle X(\omega)$ is changed to $\angle X(\omega) + \angle H(\omega)$

i.e, The input signal spectrum is modified in amplitude by a factor $|H(\omega)|$ and is shifted in phase by an angle $\angle H(\omega)$

Ex: Consider an LTI system with impulse response $h(t) = e^{-at} u(t)$. Find the response of the system to unit step input.

Sol:

$$h(t) = e^{-at} u(t) \Rightarrow H(\omega) = \frac{1}{a+j\omega}$$

$$x(t) = u(t) \Rightarrow X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

$$y(t) = x(t) * h(t)$$

Taking FT on both sides and using convolution property of FT

$$FT[y(t)] = FT[x(t) * h(t)]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$Y(\omega) = \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]\left[\frac{1}{a+j\omega}\right]$$

$$= \frac{1}{(j\omega)(a+j\omega)} + \pi \cdot \delta(\omega) \cdot \frac{1}{a+j\omega}$$

$$Y(\omega) = \frac{A}{j\omega} + \frac{B}{a+j\omega} + \frac{\pi\delta(\omega)}{a}$$ ∵ $\delta(\omega)$ exists at $\omega = 0$

here $A = \left[\frac{1}{a+j\omega}\right]_{j\omega=0} = \frac{1}{a}$

$B = \left[\frac{1}{j\omega}\right]_{j\omega=-a} = -\frac{1}{a}$

$$\therefore Y(\omega) = \frac{1}{a \cdot j\omega} - \frac{1}{a}\left[\frac{1}{a+j\omega}\right] + \frac{\pi}{a} \cdot \delta(\omega)$$

$$Y(\omega) = \frac{1}{a}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{a}\left[\frac{1}{a+j\omega}\right]$$

$$\therefore y(t) = \frac{1}{a} u(t) - \frac{1}{a} e^{-at} u(t)$$

$$y(t) = \frac{1}{a}\left[1 - e^{-at}\right] u(t) \,//$$

Ex: The output of an LTI system to an input $x(t) = e^{-2t}u(t)$ is $y(t) = e^{-t}u(t)$. Find the frequency response and impulse response of this system

sol:

$$x(t) = e^{-2t}u(t) \Rightarrow X(\omega) = \frac{1}{2+j\omega}$$

$$y(t) = e^{-t}u(t) \Rightarrow Y(\omega) = \frac{1}{1+j\omega}$$

We know

$$Y(\omega) = H(\omega)X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1+j\omega} \times 2+j\omega$$

The frequency response $\boxed{H(\omega) = \frac{2+j\omega}{1+j\omega}}$

$$1+x \overline{)\,2+x\,}(1 \qquad \frac{1+x}{1}$$

$$H(\omega) = \frac{2+j\omega}{1+j\omega} = 1 + \frac{1}{1+j\omega}$$

Taking IFT

$$\underline{\underline{h(t) = \delta(t) + e^{-t}u(t)}}$$

Ex: consider a causal LTI system with frequency response $H(\omega) = \frac{1}{3+j\omega}$. Find the input that produces the output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$.

sol:

$$H(\omega) = \frac{1}{3+j\omega}$$

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

$$Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{1}{(3+j\omega)(4+j\omega)}$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$\therefore X(\omega) = \frac{Y(\omega)}{H(\omega)} = \frac{3+j\omega}{(3+j\omega)(4+j\omega)} = \frac{1}{4+j\omega}$$

$$\therefore \underline{\underline{x(t) = e^{-4t}u(t)}} \quad //$$

Ex: Find the convolution of the following signals using FT.

$x_1(t) = e^{-3t} u(t)$ and $x_2(t) = e^{-2t} u(t)$

Sol:

Convolution property of FT says

$$y(t) = x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$

i.e, $FT[y(t)] = X_1(\omega) X_2(\omega)$

$$Y(\omega) = \frac{1}{3+j\omega} \times \frac{1}{2+j\omega} = \frac{1}{(3+j\omega)(2+j\omega)}$$

$$Y(\omega) = \frac{A}{3+j\omega} + \frac{B}{2+j\omega}$$

where $A = Y(\omega)(3+j\omega)\big|_{j\omega=-3}$

$A = \frac{1}{2+j\omega}\Big|_{j\omega=-3} = -1$

& $B = Y(\omega)(2+j\omega)\big|_{j\omega=-2}$

$B = \frac{1}{3+j\omega}\Big|_{j\omega=-2} = 1$

$$\therefore Y(\omega) = -\frac{1}{3+j\omega} + \frac{1}{2+j\omega}$$

Taking IFT

$$y(t) = -e^{-3t} u(t) + e^{-2t} u(t)$$

Ex: Find the convolution of the following signals using FT.

$x_1(t) = t e^{-t} u(t)$ and $x_2(t) = t e^{-2t} u(t)$

Ans: $y(t) = \left[-2e^{-t} + t e^{-t} + 2e^{-2t} + t e^{-2t}\right] u(t)$.

Ex: Consider a stable LTI system characterized by LCCDE

$$\frac{dy(t)}{dt} + a \cdot y(t) = x(t) \;;\; a>0$$

Find the frequency response $H(\omega)$ and impulse response $h(t)$ of the system.

sol:

$$\frac{dy(t)}{dt} + a \cdot y(t) = x(t)$$

Taking FT on both sides and using differentiation in time-domain property.

$$FT\left[\frac{dy(t)}{dt}\right] + a\, FT[y(t)] = FT[x(t)] \quad \text{—— (1)}$$

Differentiation in Time-domain property

$$\text{If} \quad y(t) \longleftrightarrow Y(\omega)$$

$$\text{then} \quad \frac{dy(t)}{dt} \longleftrightarrow (j\omega)\, Y(\omega)$$

$$\text{and} \quad \frac{d^2y(t)}{dt^2} \longleftrightarrow (j\omega)^2\, Y(\omega)$$

(1) $\Rightarrow$

$$j\omega\, Y(\omega) + a\, Y(\omega) = X(\omega)$$

$$Y(\omega)\left[a + j\omega\right] = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{a+j\omega}$$

$$\boxed{H(\omega) = \frac{1}{a+j\omega}} \quad \text{Frequency Response}$$

Taking IFT, we get impulse response

$$y(t) = e^{-at}\, u(t)$$

Ex: Find the impulse response of a stable LTI system characterized by the following LCCDE

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

sol:

Taking FT on the LCCDE, we get

$$(j\omega)^2 \cdot Y(\omega) + 4(j\omega)Y(\omega) + 3Y(\omega) = (j\omega)X(\omega) + 2X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 4(j\omega) + 3\right] = X(\omega)\left[j\omega + 2\right]$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{2+j\omega}{(j\omega)^2+4(j\omega)+3} = \frac{2+j\omega}{(j\omega+3)(j\omega+1)}$$

$$H(\omega) = \frac{A}{3+j\omega} + \frac{B}{1+j\omega}$$

where $A = H(\omega)(3+j\omega)\Big|_{j\omega=-3} = \left[\frac{2+j\omega}{1+j\omega}\right]_{j\omega=-3} = +\frac{1}{2}$

$$B = H(\omega)(1+j\omega)\Big|_{j\omega=-1} = \left[\frac{2+j\omega}{3+j\omega}\right]_{j\omega=-1} = \frac{1}{2}$$

$$H(\omega) = \frac{1}{2}\cdot\left[\frac{1}{3+j\omega}\right] + \frac{1}{2}\left[\frac{1}{1+j\omega}\right]$$

Taking IFT, we get impulse response

$$h(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

Ex: The impulse response of a causal LTI system is $h(t) = e^{-4t}u(t)$. Find the output of the system for an input $x(t) = 3e^{-t}u(t)$

sol: Hints: $y(t) = h(t) * x(t)$

$$Y(\omega) = H(\omega)\cdot X(\omega) = \frac{3}{(4+j\omega)(1+j\omega)}$$

$$Y(\omega) = \frac{A}{4+j\omega} + \frac{B}{1+j\omega} = -\frac{1}{4+j\omega} + \frac{1}{1+j\omega}$$

$$y(t) = -e^{-4t}u(t) + e^{-t}u(t)$$

Ex: The LCCDE of a causal LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Find its response to an input $x(t) = te^{-t}u(t)$.

Sol: Taking FT on the LCCDE

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega)$$

$$Y(\omega) = \left[(j\omega)^2 + 3(j\omega) + 2\right] = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3j\omega + 2} = \frac{1}{(j\omega+2)(j\omega+1)}$$

$$H(\omega) = \frac{A}{2+j\omega} + \frac{B}{1+j\omega} \qquad A = \left[\frac{1}{j\omega+1}\right]_{j\omega=-2} = -1$$

$$H(\omega) = \frac{-1}{2+j\omega} + \frac{1}{1+j\omega} \qquad B = \left[\frac{1}{j\omega+2}\right]_{j\omega=-1} = 1$$

$$h(t) = -e^{-2t}u(t) + e^{-t}u(t)$$

$$Y(\omega) = \left[\frac{1}{(j\omega)^2 + 3(j\omega) + 2}\right] X(\omega)$$

here $x(t) = t\cdot e^{-t}u(t)$

$$X(\omega) = \frac{1}{(1+j\omega)^2}$$

$$\therefore Y(\omega) = \frac{1}{(j\omega+2)(j\omega+1)(j\omega+1)^2} = \frac{1}{(1+j\omega)^3(2+j\omega)}$$

$$Y(\omega) = \frac{A_0}{(1+j\omega)^3} + \frac{A_1}{(1+j\omega)^2} + \frac{A_2}{(1+j\omega)} + \frac{B}{2+j\omega}$$

where $A_0 = (1+j\omega)^3 Y(\omega)\Big|_{j\omega=-1} = \left[\frac{1}{2+j\omega}\right]_{j\omega=-1} = 1$

$$A_1 = \frac{d}{d(j\omega)}\left[\frac{1}{2+j\omega}\right] = -\frac{1}{(2+j\omega)^2}\Bigg|_{j\omega=-1} = -1$$

$$A_2 = \frac{1}{2!}\frac{d^2}{dj\omega^2}\left[\frac{1}{2+j\omega}\right]_{j\omega=-1} = \frac{1}{2}\frac{d}{dj\omega}\left[-\frac{1}{(2+j\omega)^2}\right]_{j\omega=-1}$$

$$= \left[\frac{1}{2} \times -(-2)\cdot(2+j\omega)^{-3}\right]_{j\omega=-1} = \underline{\underline{1}}$$

$$B = \left[(2+j\omega)Y(\omega)\right]_{j\omega=-2} = \left[\frac{1}{(1+j\omega)^3}\right]_{j\omega=-2} = -1$$

$$\therefore\ Y(\omega) = \frac{1}{(1+j\omega)^3} - \frac{1}{(1+j\omega)^2} + \frac{1}{(1+j\omega)} - \frac{1}{(2+j\omega)}$$

Taking IFT, we get output $y(t)$

$$y(t) = \frac{t^2}{2}e^{-t}u(t) - t\cdot e^{-t}u(t) + e^{-t}u(t) - e^{-2t}u(t)$$

Note:
$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$
$$t\cdot e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$$
$$\frac{t^2}{2}\cdot e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^3}$$

Ex: The frequency response of an LTI system is $H(\omega) = \frac{4}{2+j2\omega}$. Find its response to $x(t) = 3\sin(2t)$.

Sol: $H(\omega) = \frac{4}{2+j2\omega}$ and $x(t) = 3\sin(2t)$

The i/p signal frequency is 2 rad/s. Hence frequency response of system at $\omega=2$ is $H(\omega) = \frac{4}{2+j2(2)} = \frac{4}{2+j4}$

$$H(\omega) = 1.414\angle{-45^\circ}$$

i.e., the system modifies the input signal amplitude by 1.414 times and shifts its phase by -45°.

$$\therefore\ y(t) = 1.414\left[3\sin(2t-45^\circ)\right]$$

$$\underline{y(t) = 4.242\sin(2t-45^\circ)}$$